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Total No. of Questions : 09] [Total Pages : 05

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M.A./M.Sc. EXAMINATION, 2025

(Third Semester)

MATHEMATICS

M-305A

Analytical Number Theory

(Elective-II)

Time : 3 *Hours*] [*Maximum Marks* : 80

The candidates shall limit their answer precisely within the answer-book (40 pages) issued to them and no supplementary/continuation sheet will be issued.

Note : Attempt *Five* questions in all, selecting at least one question (but not more than *two* questions) from each Section. All questions carry equal marks.

Section I

1. (a) Show that the cube of any integer is of the form $7k$ or $7k + 1$. **8**

   (b) If a and b are integers, not both zero, then prove that a and b are relatively prime if and only if there exist integers x and y such that $1 = ax + by$. **8**

2. (a) State and prove fundamental theorem of arithmetic. **10**

   (b) If $n > 2$, prove that there exist a prime p satisfying $n < p < n!$. **6**

3. (a) If $P(x) = \sum_{k=0}^{m} c_k x^k$ is a polynomial function of x with integral coefficients c_k and $a \equiv b \pmod{n}$, then prove that $P(a) \equiv P(b) \pmod{n}$. **8**

(b) Solve the simultaneous congruences $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 5$, $x \equiv 3 \pmod 7$. **8**

Section II

4. (a) If p and q are distinct primes with $a^p \equiv a \pmod q$ and $a^q \equiv a \pmod p$, prove that $a^{pq} = a \pmod{pq}$. **8**

(b) Using Fermat's theorem verify that 17 divides $11^{104} + 1$. **8**

5. (a) If f is a multiplicative function and F is defined by $F(n) = \sum_{d|n} f(d)$, prove that F is also multiplicative. **8**

(b) Find the highest power of 5 dividing 1000! and the highest power of 7 dividing 2000 !. **8**

6. (a) If p is a prime and $k > 0$, prove that

$$\varphi(p^k) = p^k - p^{k-1} = p^k\left(1-\frac{1}{p}\right).$$ **8**

(b) If $n \geq 1$ and $\gcd(a, n) = 1$, prove that

$a^{\varphi(n)} \equiv 1 (\text{mod } n)$. **8**

Section III

7. (a) If the integer a has order k modulo n and $h > 0$, prove that a^h has order $k/gcd(h, k)$ modulo n. **8**

(b) If p is a prime and $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, $a_n \not\equiv 0 \ (mod\, p)$ is a polynomial of degree $n \geq 1$ with integral coefficients, then prove that the congruence $f(x) \equiv 0 (\text{mod } p)$ has at most n incongruent solutions modulo p. **8**

8. (a) Find the index of 5 relative to each of the primitive roots of 13. **8**

(b) Find the quadratic residues of 29 and 31. **8**

9. (a) If p is an odd prime, prove that $\sum_{a=1}^{p-1}(a/p)=0$, there are precisely $\frac{(p-1)}{2}$ quadratic residues and $\frac{(p-1)}{2}$ quadratic non-residues of p. **8**

(b) Solve the congruence $x^2 \equiv 31 (\text{mod } 11^4)$. **8**

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